TRACK DEFLECTION ANALYSIS

Nikola Svobodová*,1

*Nikola.Svobodova@vut.cz
1Institute of Railway Structures and Constructions, Faculty of Civil Engineering, Brno University of Technology, Veveří 331/95, 602 00 Brno, Czech Republic

Abstract

This review paper discusses various analytical models that can describe the vehicle-track system without requiring extensive and computationally demanding numerical models. All these analytical models aim to capture the behaviour of the track structure as a rail vehicle passes over it as accurately as possible, each in its unique way, depending on the assumptions they are based on. The objective of this paper is to evaluate their potential and limitations from different perspectives, such as computational simplicity, result availability, solution accuracy, and variability in solution forms.

Keywords

Track deflection, Elastic modulus, Dynamic loading, Static loading, Analytical model

1 INTRODUCTION

Given the current trend of increasing the design speed of railway vehicles, the overall applied load must be considered, as it inevitably increases due to its dynamic component [1], despite the evident efforts to reduce the weight of train sets [2].

The total load on railway structures consists of both static and dynamic components. Static loads are associated with the self-weight of the structure and stationary loads from train sets. Dynamic loads result from the interaction between the wheels of a moving train and the rail [1].

As the load on railway structures grows, so do the requirements for their resistance to stress, stability, durability, and safety. The track structure must provide adequate support for the movement of railway vehicles without permanent deformations that could negatively impact any of the aforementioned criteria [3].

It is essential to understand the physical processes occurring during the movement of railway vehicles, making track deflection analysis a valuable tool to design railway structures effectively.

This paper aims to provide an overview of available analytical models for analysing and predicting track deflection. These models offer insight into the physical nature of the phenomenon, each adopting a different perspective and utilizing distinct or similar tools and principles. The paper does not present new experimental data but synthesizes and consolidates existing findings. It primarily focuses on classical and advanced contact models, summarizing their assumptions, advantages, and limitations. The comparison of these methodologies can serve as a basis for selecting an appropriate approach to railway track design that meets the increasing demands of the modern era.

2 METHODOLOGY

Two fundamental concepts emerge in geotechnics: active loading and passive resistance. Active loading results from the application of actual external forces on a foundation structure. Passive resistance, in turn, is the reaction to these external forces and manifests through deformation. There are two primary approaches in geotechnics for determining passive resistance [4].

The first approach is the elastic half-space. This concept is based on the fundamental assumptions of elasticity theory and is intuitively the most natural choice. However, apart from the model case of a single force acting on

Received 03/03/2025/Accepted 26/03/2025/Published 30/09/2025 DOI 10.13164/juniorstav.2025.25052

a homogeneous isotropic medium, as described by Boussinesq, this approach becomes computationally very complex for more intricate cases, making it challenging to obtain a solution [4].

The second major approach for determining passive resistance is contact model, which stem from entirely different considerations. A contact model fundamentally concentrates the properties of the subsoil into the contact interface between the structure and the subgrade. The most common and oldest contact model is the Winkler model, which, however, neglects shear cohesion. Other contact models build upon the original model by incorporating various assumptions. For instance, the Pasternak model includes a shear component, thus addressing a key limitation of the primary model [4].

There are also different strategies for modelling. The first approach involves a static analysis, where the model assumes that the load is applied slowly and uniformly or remains entirely constant, with no inertial forces generated. The second approach is more complex, extending the static model by incorporating dynamic elements, meaning it accounts for loading that varies over time and includes the effects of inertial and damping forces [5].

Static models

Timoshenko model (Zimmermann model) using the Winkler foundation model

Winkler was the first to describe and analyse the interaction between foundations and subgrade soil as an elastic beam on an elastic foundation. In his model, the subgrade behaves as a system of independent springs. This model is simplified by neglecting the effects of horizontal forces and deforming only in the region directly under the applied load, meaning no deflection basin is formed [3].

In railway construction, the Winkler foundation model can be envisioned as an infinitely long beam resting on an elastic foundation - Timoshenko's model of a rod in elastic confinement. The beam represents the rail, characterized by bending stiffness, while the elastic foundation represents the subgrade, expressed by stiffness k. The rail is subjected to a stationary wheel force Q at a given location x. The response to this load manifests as both rail deflection and subgrade compression. This is an analytical, statically indeterminate problem, meaning there are more unknowns than the number of equilibrium equations available. Therefore, deformation conditions must also be used. The solution involves determining equilibrium equations and using the bending equation. In railway construction, Timoshenko's model is also referred to as the Zimmermann model. The solution is derived from the differential equation (1) [1], [6].

$$EI \cdot \frac{d^4 w(x)}{dx^4} + k \cdot w(x) = 0 \tag{1}$$

where EI is the flexural rigidity of the rail in Nm², $w(x)$ is the vertical deflection in m, x is the horizontal coordinate in m, and k is the Winkler modulus of the elastic foundation in N/m², and parameter k describes the stiffness of the subgrade.

Pasternak foundation model

The Winkler foundation model does not account for the shear cohesion of the soil. The foundation deforms only in the area directly under the load, without forming a deflection basin. As a result, it is not possible to determine the extent of the effects on surrounding structures [3]. In railway construction, which Timoshenko studied, the continuity of deformations due to track loading is ensured by the stiffness of the track structure, which rests on the subgrade. Timoshenko likely did not see a need to introduce assumptions that would enforce interconnection between soil points [1]. Another consequence of the absence of shear interaction is that the predicted deflection is greater than what actually occurs [2]. The Pasternak foundation model, which is derived from the Winkler model by introducing a shear interaction term within the soil, addresses some of the aforementioned limitations and provides a more accurate representation of the behaviour of the soil medium under loading [3]. The model considers an infinitely long rail subjected to a wheel force Q. The rail is characterized by bending stiffness EI and is rigidly connected to a shear element, which deforms only under the influence of transverse shear forces. This shear element is placed on an elastic foundation, representing the subgrade. The solution process is similar to that of the Timoshenko model. Due to the number of unknowns exceeding the number of equilibrium equations, the bending equation must be employed [1], [5]. The solution is derived from the differential equation (2).

$$EI \cdot \frac{d^4 w(x)}{dx^4} - GA \cdot \frac{d^2 w}{dx^2} + k \cdot w(x) = 0 \tag{2}$$

where *EI* is the flexural rigidity of the rail in Nm², *w(x)* is the vertical deflection in m, *x* is the horizontal coordinate in m, *k* is the Winkler modulus, which represents the stiffness of the subgrade, in N/m², and *GA* is the shear stiffness of the foundation in N/m.

Static system of a two-layer model

The two-layer system model refines the Timoshenko (Zimmermann) model by extending an additional layer. This enhancement enables the investigation of rail pads' behaviour or the assessment of the subgrade. The first layer of the model can be interpreted, as in previous models, as an infinitely long rail. The second layer represents the rail supports, which may consist of transverse sleepers or a fixed track system [5]. The solution is derived from the differential equation (3) and equation (4).

$$EI_1 \cdot \frac{d^4 w_1(x)}{dx^4} + k_1 \cdot [w_1(x) - w_2(x)] = 0 \tag{3}$$

$$EI_2 \cdot \frac{d^4 w_2(x)}{dx^4} + (k_1 + k_2) \cdot w_2(x) - k_1 \cdot w_1(x) = 0 \tag{4}$$

where EI_1 and EI_2 are the flexural rigidities of the first and second layers in Nm², specifically, while EI_1 describes the bending behaviour of the rail, EI_2 is typically small and captures the residual flexibility of the sleeper layer, $w_1(x,t)$ and $w_2(x,t)$ are the vertical deflections of the first and second layers in m, k_1 and k_2 are the elastic coefficients in N/m², specifically the stiffness k_1 reflects the elasticity of rail fasteners, and k_2 represents the deformability of the subgrade, and finally *x* is the horizontal coordinate in m.

Dynamic models

Frýba model (Timoshenko model for a dynamic system)

The Frýba model is based on the Timoshenko (Zimmermann) concept of a beam on a Winkler elastic foundation, but it differs by incorporating a moving load, whose effects induce a dynamic response. Frýba included the influence of inertia and damping in the model. As a result, the infinitely long rail is characterized both by its bending stiffness *EI* and its mass *m* [7]. The solution is derived from the differential equation (5).

$$EI \cdot \frac{d^4 w(x,t)}{dx^4} + m \cdot \frac{d^2 w(x,t)}{dt^2} + c \cdot \frac{dw(x,t)}{dt} + k \cdot w(x,t) = 0 \tag{5}$$

where *EI* is the flexural rigidity of the rail in Nm², *w(x,t)* is the vertical deflection in m, *x* is the horizontal coordinate in m, *t* is the time in s, *m* is the mass per unit length of the rail in kg/m, *c* is the damping coefficient (reflects energy dissipation in the track system) in Ns/m², *k* is the Winkler modulus in N/m² and describes the stiffness of the subgrade.

Pasternak foundation model – dynamic system

The model was developed by extending Frýba's calculation with an element that transfers shear loading. It thus represents a Pasternak foundation model subjected to a moving force [5]. The solution is derived from the differential equation (6).

$$EI \cdot \frac{d^4 w(x,t)}{dx^4} + m \cdot \frac{d^2 w(x,t)}{dt^2} + c \cdot \frac{dw(x,t)}{dt} - GA \cdot \frac{d^2 w(x,t)}{dx^2} + k \cdot w(x,t) = 0 \tag{6}$$

where *EI* is the flexural rigidity of the rail in Nm², *w(x,t)* is the vertical deflection in m, *x* is the horizontal coordinate in m, *t* is the time in s, *m* is the mass per unit length of the rail in kg/m, *c* is the damping coefficient (reflects energy dissipation in the track system) in Ns/m², *k* is the Winkler modulus in N/m² and describes the stiffness of the subgrade, *GA* is the shear stiffness of the foundation in N/m.

Two-layer system – dynamic system, Winkler foundation model

This model is based on the previously mentioned two-layer system model, but it differs by considering the moving nature of the applied load [5]. The solution is derived from the differential equation (7) and equation (8).

DOI 10.13164/juniorstav.2025.25052

$$EI_1 \cdot \frac{d^4 w_1(x,t)}{dx^4} + m_1 \cdot \frac{d^2 w_1(x,t)}{dt^2} + c_1 \cdot \frac{dw_1(x,t)}{dt} + k_1 \cdot (w_1(x,t) - w_2(x,t)) = 0 \tag{7}$$

$$EI_2 \cdot \frac{d^4 w_2(x,t)}{dx^4} + m_2 \cdot \frac{d^2 w_2(x,t)}{dt^2} + c_2 \cdot \frac{dw_2(x,t)}{dt} + (k_1 + k_2) \cdot w_2(x,t) - k_1 \cdot w_1(x,t) = 0 \tag{8}$$

where EI_1 and EI_2 are the flexural rigidities of the first and second layers in Nm², specifically, while EI_1 describes the bending behaviour of the rail, EI_2 is typically small and captures the residual flexibility of the sleeper layer, $w_1(x,t)$ and $w_2(x,t)$ are the vertical deflections of the first and second layers in m, m_1 is mass per unit length of the upper layer (rail + fasteners) in kg/m, m_1 is mass per unit length of the lower layer (sleepers or underlying mass) in kg/m, c_1 is damping coefficient between upper and lower layer (fastener damping) in Ns/m² and c_2 is damping coefficient of the subgrade in Ns/m², k_1 and k_2 are the elastic coefficients in N/m², specifically the stiffness k_1 reflects the elasticity of rail fasteners, and k_2 represents the deformability of the subgrade, t is the time in s, and finally x is the horizontal coordinate in m.

Two-layer system – dynamic system, Pasternak foundation model

This model builds on the previous two-layer dynamic system. In addition to the effects of moving loads, shear interaction is also considered. The fundamental assumptions of an infinitely long beam on an elastic foundation remain valid [5]. The solution is derived from the differential equation (9) and equation (10).

$$EI_1 \cdot \frac{d^4 w_1(x,t)}{dx^4} + m_1 \cdot \frac{d^2 w_1(x,t)}{dt^2} + c_1 \cdot \frac{dw_1(x,t)}{dt} + k_1 \cdot (w_1(x,t) - w_2(x,t)) = 0 \tag{9}$$

$$EI_2 \cdot \frac{d^4 w_2(x,t)}{dx^4} - GA \cdot \frac{d^2 w(x,t)}{dx^2} + m_2 \cdot \frac{d^2 w_2(x,t)}{dt^2} + c_2 \cdot \frac{dw_2(x,t)}{dt} + (k_1 + k_2) \cdot w_2(x,t) - k_1 \cdot w_1 = 0 \tag{10}$$

where EI_1 and EI_2 are the flexural rigidities of the first and second layers in Nm², specifically, while EI_1 describes the bending behaviour of the rail, EI_2 is typically small and captures the residual flexibility of the sleeper layer, $w_1(x,t)$ and $w_2(x,t)$ is the vertical deflections of the first and second layers in m, m_1 is mass per unit length of the upper layer (rail + fasteners) in kg/m, m_1 is mass per unit length of the lower layer (sleepers or underlying mass) in kg/m, c_1 is damping coefficient between upper and lower layer (fastener damping) in Ns/m² and c_2 is damping coefficient of the subgrade in Ns/m², k_1 and k_2 are the elastic coefficients in N/m², specifically the stiffness k_1 reflects the elasticity of rail fasteners, and k_2 represents the deformability of the subgrade, GA is the shear stiffness of the second layer in N/m and t is the time in s, and finally x is the horizontal coordinate in m.

3 RESULTS

The following Tab. 1 is derived from the methodology and summarizes the components of the individual models.

Tab. 1 Components of individual models.

Model	Bending	Shear	Dynamics	Number of layers
Timoshenko (Zimmermann)	Yes	No	No	*1*
Pasternak	Yes	Yes	No	*1*
Static two-layer model	Yes	No	No	*2*
Frýba	Yes	No	Yes	*1*
Pasternak – dynamic system	Yes	Yes	Yes	*1*
Two-layer dynamic model	Yes	No	Yes	*2*
Two-layer dynamic model with shear	Yes	Yes	Yes	*2*

The following Tab. 2 summarizes examples of possible analytical solutions [1], [5], [8]. For solutions that lead to partial differential equations (PDEs) or ordinary differential equations (ODEs), the number of equations and

DOI 10.13164/juniorstav.2025.25052

unknowns remaining at the final stage of the calculation is provided in more detail [5]. The description of individual models is derived from the methodology and the complexity of the solution.

Tab. 2 Comparison of individual analytical models.

Model	Examples of analytical solutions	ODE/PDE	Description
Timoshenko (Zimmermann)	**ODE**	The ODE solution consists of 4 equations with 4 unknowns, some of which drop out after applying boundary conditions.	Basic vehicle–track model, simple solution, fast and accessible results.
Pasternak	**ODE**	The ODE solution consists of 4 equations with 4 unknowns, some of which drop out after applying boundary conditions.	Vehicle–track model with shear interaction. One of the simpler solutions is relatively fast and accessible results.
Static two-layer model	**ODE system**	The ODE solution consists of 4 equations with 4 unknowns.	Vehicle–track model with an additional layer, without considering shear. More complex calculation.
Frýba	**PDE** Method of variable separation Fourier transform Laplace transform	The PDE solution after variable separation is a system of 4 equations with 4 unknowns.	Vehicle–track model incorporating dynamics. More complex calculation.
Pasternak – dynamic system	**PDE** Method of variable separation Fourier transform Laplace transform	The PDE solution after variable separation results in a system of 4 equations with 4 unknowns.	Vehicle–track model incorporating shear and dynamics. More complex calculation.
Two-layer dynamic model	**PDE system** Method of variable separation Fourier transform Laplace transform	The PDE solution after decomposition into characteristic roots results in 8 equations with 8 unknowns.	Vehicle–track model incorporating an additional layer and dynamic behaviour. Complex calculation.
Two-layer dynamic model with shear	**PDE system** Fourier transform Laplace transform	The PDE solution after decomposition into characteristic roots results in 8 equations with 8 unknowns.	Vehicle–track model incorporating shear, an additional layer, and dynamic behaviour. Complex calculation.

4 DISCUSSION

The analytical solution of each beam model depends on the complexity of their differential equations. These equations may include various components such as bending, shear, damping, and inertia, or be expressed with multiple layers and time dependency.

DOI 10.13164/juniorstav.2025.25052

The simplest models, such as the Timoshenko model or the Pasternak foundation model, naturally lead to solutions using ordinary differential equations (ODEs). Their results can be obtained relatively quickly, even without computational software. Both models are suitable for obtaining initial estimates before using more complex models or for calculating track deflection for slow-moving railway vehicles, where dynamic effects do not play a significant role. The Pasternak foundation model additionally considers shear interaction, providing a more realistic representation of track behaviour. Although computationally more demanding than the Winkler model, it can still be solved without computational software by applying an appropriate substitution.

The static two-layer system extends the Timoshenko model by adding a second layer, which can represent rail supports, such as sleepers. By considering two interacting layers, this model more accurately captures force transmission between the rail and subgrade, making it a closer approximation of reality compared to the Timoshenko model, where the subgrade is modelled as independent springs. This approach allows for an assessment of sleeper distribution effects on track deflection, which is logically absent in the Timoshenko model. However, for higher-speed railway vehicles, it is necessary to include dynamic behaviour, as a purely static approach may not provide sufficiently accurate results. Additionally, this model neglects shear interaction, which could affect deformation accuracy.

The Frýba model extends the original beam model by incorporating dynamic behaviour and time dependency. The Frýba beam problem can be formulated as a partial differential equation (PDE) to determine the instantaneous response to loading. Even when applying simplified boundary conditions, this remains a complex, asymmetric problem, which may require numerical solutions for more complicated cases. The Frýba model is also useful for frequency response analysis, and it can be solved using Laplace and Fourier transforms. However, these transformations shift the problem to the frequency domain, meaning they do not directly provide track deflection results, but rather offer insights into how the rail responds to different loading frequencies (Fourier transform) or how it reacts to an impulse (Laplace transform).

The Pasternak foundation model with dynamic behaviour can be considered an equivalent of the Timoshenko model but without the rotational inertia of the cross-section. Like the Frýba model, it can be analysed using Fourier and Laplace transforms, allowing for frequency domain analysis as described earlier. The standard approach for determining track deflection over time involves solving a partial differential equation (PDE), which is computationally demanding due to the inclusion of both dynamic effects and shear deformation. However, this provides a more realistic representation of track behaviour under loading, obtaining more accurate results compared to basic models.

The most computationally demanding, yet also the most accurate models, are the two-layer dynamic model and its extended version with shear interaction. Solving the partial differential equations (PDEs) results in a system of eight equations with eight unknowns, which may require computational software. Like previous models, Laplace and Fourier transforms can be used for analysis, but they do not provide a direct solution for instantaneous track deflection. Instead, they enable an examination of the rail's response to load in the frequency domain.

It is important to emphasize that the computational complexity of these analytical models differs fundamentally from that of numerical methods. Since the models are based on differential equations, the primary challenge lies in the correct formulation and transformation of these equations rather than in solving them computationally. Once the equations are determined and can be implemented in a MATLAB environment, the solution itself can often be obtained within a few seconds. Most of the time required is thus spent on setting up the model, particularly in the case of more complex dynamic or multilayer systems. Once automated, the computational process becomes very efficient, making analytical modelling well-suited for practical use.

Practical examples of applying analytical models can be found in both the study by Kulich and Plášek (2022) and the diploma thesis by Kulich (2017), both of which are based on measurements conducted on the railway track in Planá nad Lužnicí. The superstructure at the site consists of 60 E1 rails, W14 fastening systems, and B 91S/1 concrete sleepers.

In the aforementioned article, a two-layer dynamic model with Pasternak foundation was used for reverse analysis - that is, for the back-calculation of model input parameters to match the measured rail deflection under a passing passenger train. The observed rail deflection was approximately 0.6 mm, and the optimized model captured the deformation shape very well [9].

In the diploma thesis, several analytical track models were compared under the load of a two-axle bogie of a traction vehicle travelling at a speed of 77 km/h. The measured deflections ranged between 0.9 and 1.0 mm. Among the simpler models, the Frýba model showed the best agreement with the measurements. However, the most accurate results were provided by the two-layer dynamic model with a moving load and shear interaction.

This model successfully captured both the peak deflection and the deformation behaviour between axles. These examples confirm that analytical models—when properly set up and calibrated—can realistically represent the behaviour of railway structures in practice [5].

5 CONCLUSION

The article discussed various analytical models for railway track deflection, evaluating their advantages and limitations, and highlighting their suitability for different scenarios. Contact models are representative of the vehicle–rail interaction and are more appropriate for determining track deflection under railway vehicle loading.

Advanced models that incorporate dynamic behaviour provide a more realistic representation of railway track behaviour under moving loads compared to static contact models.

Future research could focus on numerical models, the combination of analytical and numerical approaches, as well as integrating computational models with experimental data and machine learning or using experimental data combined with machine learning to enhance the prediction of structural response to loading.

From a practical perspective, the choice of a specific analytical model depends on the design stage, the expected train speed, and the required level of detail in the structural response assessment. The Timoshenko and Pasternak models are suitable for designing tracks with lower operational speeds or in sections where dynamic effects can be neglected. The static two-layer model provides a better understanding of the interaction between the rail and the sleepers and is therefore appropriate for designs involving fastener behaviour or sleeper spacing. For high-speed lines or transition zones, it is necessary to employ dynamic models, such as the Frýba model or dynamic variants of the Pasternak model, which account for inertial and damping effects. The most complex models, such as two-layer dynamic systems with shear interaction, are better suited for detailed analyses and research purposes, where accuracy is prioritized over simplicity.

References

[1] ESVELD, C. Modern Railway Track. Second edition; MRT-Production, 2001, ISBN 90- 8003243-3

[2] VOPÁLENSKÁ, M. VOPÁLENSKÁ, M. Innovative solutions in the railway industry for sustainable mobility. Scientific and Technical Journal, 2019, p. 8. Railway Infrastructure Administration, state organization. Available from: https://www.spravazeleznic.cz/documents/50004227/87152001/Inovativn%C3%AD%2B%C5%99e%C5%A1en%C3%AD%2B%C5%BEelezni%C4%8Dn%C3%ADho%2Bpr%C5%AFmyslu%2Bpro%2Budr%C5%BEitelnou%2Bmobilitu.pdf/f4079d08-ff8d-41b5-940f-698ef9377ffd

[3] IŽVOLT, Libor. Railway substructure: loading, diagnostics, design and construction of substructure layers. Žilina: University of Žilina, 2008.

[4] BROŽOVSKÝ, J. a MATERNA, A. Fundamentals of the Mathematical Theory of Elasticity. Ostrava: VŠB – Technical University of Ostrava and University of West Bohemia in Pilsen, 2012. 213 p.

[5] KULICH, Pavel. Dynamic analysis of railway track. Brno, 2017. 46 pages + 22 pages of appendices. Diploma thesis. Brno University of Technology, Faculty of Civil Engineering, Institute of Railway Structures and Constructions. Supervisor: Assoc. Prof. Ing. Otto Plášek, Ph.D.

[6] TIMOSHENKO, S. Method of analysis of statical and dynamical stresses in rail. Proc. Second Int. Congress of Appl. Mech., Zurich, Switzerland, 1926, pp. 12-17.

[7] FRÝBA, Ladislav. Vibration of solids and structures under moving loads. Prague: Academia, 1972

[8] REKTORYS, K. a KOLEKTIV. Přehled užité matematiky, SNTL, Prague, 1963

[9] KULICH, P. and O. PLÁŠEK. Railway track deflection analysis by using evolutionary algorithms. Acta Polytechnica CTU Proceedings, vol. 35, pp. 23–26, 2022. DOI: 10.14311/APP.2022.35.0023

DOI 10.13164/juniorstav.2025.25052