ADVANCED ANALYTICAL MODEL OF STRENGTH IN QUASIBRITTLE MEDIA

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Abstract
The purpose of this work is to extend the use of a recently developed analytical model of concrete fracture behavior to general load cases. The model reflects the stress redistribution processes in the fracture process zone by averaging the local material strength. Analytically, the probability that the spatial stress field attains the averaged strength field is obtained, resulting in a full probability density function of random structural strength, which naturally captures all sources of structural size effect. The expanded version of this model can be used in various general load cases.

Keywords
Random strength field, mesoscale, concrete, FPZ, weibull theory

1 INTRODUCTION

The basic behavior of structural materials, especially heterogeneous materials exhibiting quasi-brittle behavior like concrete, ceramics and some geological materials, includes a dependence of certain properties on structural size. This behavior has been studied for several decades, revealing two fundamental sources: the statistical source, which is based on randomness in material properties over the *specimen* volume, and the energetic source, which is based on different ways of energy dissipation in specimens of different sizes. This discrepancy arises from the different ratio between the volume of the specimen and the volume defined as the fracture process zone (FPZ), where strain and stress relations are highly nonlinear and fracture energy dissipation takes place.

One way to predict this property is through the use of numerical simulations, such as finite element simulations with localization limiters, regularized models or discrete models where displacement over the specimen is not continuous [1], [2], [3]. However, the issue with this type of numerical simulation is the high computational cost. For this reason, it is useful to develop analytical approaches based on theoretical principles.

For the energetic sources of size effect, there is a famous series of work by Bažant [4], [5]. Regarding the statistical source of size effect, the analytical model was originally proposed by Weibull [6] where the structure is simplified as a group of volumes connected in series, leading to an exponential law. However, this approach relies on some strong presumptions: the structure failure is considered to occur in one infinitely small volume and there is no spatial correlation of material properties. These presumptions are well known for not being realistic in real concrete structures.

For this reason, other advanced analytical models for statistical size effect were 1eveloper, such as Ditlevsen [7], [8], which considers material properties as a random process, and by probabilistic calculations it predicts the mean state of stress distribution which is met by material strength. However, a limitation of the Ditlevsen model is the fact that it inherits one of the strong presumptions of the Weibull model – the structure failure is still considered as developing in one infinitely small point, where the stress and strength function meet.

Work objective

In real structures, when the strength capacity in one infinite point is reached, stress is simply redistributed to its neighbors. If the volume of material where strength capacity is exceeded reaches some critical volume (with is proportional to the FPZ), the stress can't be distributed anymore, and the structure will fail. The recently developed model by Vořechovský and Eliáš [9] builds on Ditlevsen's work by incorporating the idea of averaging stress and strength fields by such critical volume. If this new averaged stress and strength fields meet, it means the strength capacity is reached in the whole critical volume, and the structure will fail. However, this model was primarily developed for some notched and unnotched bending load cases and the stress field was considered a constant function for simplification.

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The objective of this work is therefore to extent the mentioned model to general load cases and study the effects of different loadings on size effect law.

2 METHODOLOGY

Basic terminology

Let's consider a general concrete specimen where a position in the specimen is described by a vector of coordinates $\mathbf{x}$. We consider material properties to be dependent on their position on the specimen, so we can write the tensile strength and fracture energy of concrete as functions $f_t(\mathbf{x})$ and $G_t(\mathbf{x})$. These material parameters have a random character over the specimen volume, so we consider them as $f_t(\mathbf{x}) = H(\mathbf{x})\, f_{t,0}$, $G_t(\mathbf{x}) = H(\mathbf{x})\, G_{t,0}$, where $f_{t,0}$, $G_{t,0}$ are mean values of material parameters, and $H(\mathbf{x})$ is a random field which, for every coordinate $\mathbf{x}$, gives a random value from a Gaussian distribution with a mean value of $\mu_h = 1$ and a given standard deviation δ_h. Because neighboring material points must have similar material properties, $H(\mathbf{x})$ in two given points $\mathbf{x}_\mathrm{i}$ and $\mathbf{x}_\mathrm{j}$ must be spatially correlated. This special correlation is given by the autocorrelation function $\rho(\mathbf{x}_\mathrm{i}, \mathbf{x}_\mathrm{j}, l_\rho) = \exp[-(\|\mathbf{x}_\mathrm{i} - \mathbf{x}_\mathrm{j}\|/l_\rho)^2]$, where l_ρ is a driving parameter known as the autocorrelation length. With the parameter l_ρ, we can control "how much" material parameters will vary over the volume. The average tensile strength for a given critical volume is defined as $f_{t,eff}(\mathbf{x}) = H_{eff}(\mathbf{x})\, f_{t,0}$, where $H_{eff}(\mathbf{x})$ stands for the random field averaged over the critical volume.

Besides the strength field, we consider a spatial function of a given stress measure $\sigma(\mathbf{x})$ and the average of this stress measure over the critical volume is defined as the effective stress measure $\sigma_{eff}(\mathbf{x})$.

Averaging process

For a 1D specimen, where x is a given point and T represents the length of the averaging critical volume, the average random field (referred to as a random process in 1D) H_{eff} (x) is then defined as integral (1):

$$H(\mathrm{x}, T) = \frac{1}{T}\int_{x-T/2}^{x+T/2} H_{eff}(\mathrm{x})dx \tag{1}$$

The averaging has two significant effects on the random process. It reduces its variation as follows, where $\gamma(T)$:

$$\delta_{H,eff}^2 = \gamma(T)\,\delta_H^2 \tag{2}$$

$$\gamma(T, l_\rho) = \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right)\rho(\tau)\; d\tau \tag{3}$$

is the variation function and it affects the correlation properties of the field. According to Vanmarcke [10], the product of $\left(\frac{\delta}{\mu}\right)^2\theta$, where θ is the scale of fluctuation defined as double the area under the autocorrelation function, must remain invariant. As a result of that, the autocorrelation function of the averaged random process yields (4):

$$\rho_{eff}(\tau, T, l_\rho) = \frac{1}{2T^2\gamma(T)}\,[(T+\tau)^2\,\gamma(T+\tau) - 2\tau^2\gamma(\tau) + (T-\tau)^2\,\gamma(T-\tau)] \tag{4}$$

For the 3D case with a rectangular critical volume $T_1\ T_2\ T_3$, the reduction of variance is given in (5), (6):

$$\delta_{H,eff}^2 = \gamma(T_1, T_2, T_3)\,\delta_H^2 \tag{5}$$

$$\gamma(T_1, T_2, T_3) = \gamma(T_1)\,\gamma(T_2)\gamma(T_3) \tag{6}$$

where $\gamma(T_1, T_2, T_3)$ is the variance function of the whole critical volume $T_1\ T_2\ T_3$.

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Probability of averaged stress and strength crossing

According to Ditlevsen [7], [8], the probability that a random process will uncross the function $f(\mathrm{x})$ on the interval L is given as follows (7), (8), (9):

$$P\left(H(\mathrm{x})_{max} > f\ (\mathrm{x})\right) = \Phi(\sigma(\mathrm{x})\exp\left[-\sqrt{2}\pi\ I_1 + I_2\ \right] \tag{7}$$

$$I_1 = \int_0^{\mathrm{L}} \lambda\ \phi\left(\frac{f\ ^{\prime}(\mathrm{x})}{\lambda\ \sqrt{2}\pi}\right)\ \frac{\phi\ f(\mathrm{x})}{\Phi\ f(\mathrm{x})} dx \tag{8}$$

$$I_2 = \int_0^{\mathrm{L}} \Phi\left(-\frac{f\ ^{\prime}(\mathrm{x})}{\lambda\ \sqrt{2}\pi}\right)\ \frac{\phi\ f(\mathrm{x})}{\Phi\ f(\mathrm{x})} f^{\prime}(x) dx \tag{9}$$

For the random field averaged in one direction, the coefficient λ is given by (10):

$$\lambda = \frac{1}{T_1}\sqrt{(1 - \rho\left(T_1;\ l_\rho\right)\ )}\ /\ \pi\ \gamma(T_1) \tag{10}$$

From that, we can simply evaluate the probability of structure failure for a given stress in (11):

$$G_f\left(\sigma(\mathrm{x})\right) = P(f_t(x) < \sigma\ (\mathrm{x})) = -\ P\ \left(H(\mathrm{x})_{max}\ > \frac{(\sigma\ (\mathrm{x}) - f_0)}{\delta_h}\right) \tag{11}$$

This gives the probability that stress in any place of the structural element of length L would exceed the tensile strength function $f_t(\mathrm{x})$.

The last step is to predict the stress level $\sigma\ (\mathrm{x})$, where the structure fails on average. This is simply done by considering the stress level as $\sigma\ (\mathrm{x}) = k\ \sigma_0$ where σ_0 is the unit stress and k is the scaling parameter. Then, we substitute into equation (11) and integrate over k to find the mean scaling parameter

$$k_{mean} = \int_{-\infty}^{\infty} k\ dG_f(k\sigma_0) \tag{11}$$

From the mean scaling parameter k_{mean}, you can calculate the mean maximum load and mean nominal strength.

3 RESULTS

Let's show the model behavior using a one-dimensional beam of length L loaded by three different load cases: a beam loaded by pure tension with a constant stress level over its length, a beam loaded by eccentric tension with a linear stress distribution and a beam loaded with a distributed load, where the stress from the bending moment is parabolic. Load cases are shown in Fig. 1.

For each case, the input parameters apart from the stress function are the same:

- $L = 2$ m,
- $f_{0,} =\ 3$ MPa,
- $\delta_h = 0.3$,
- $l_\rho\ = 100$ mm.

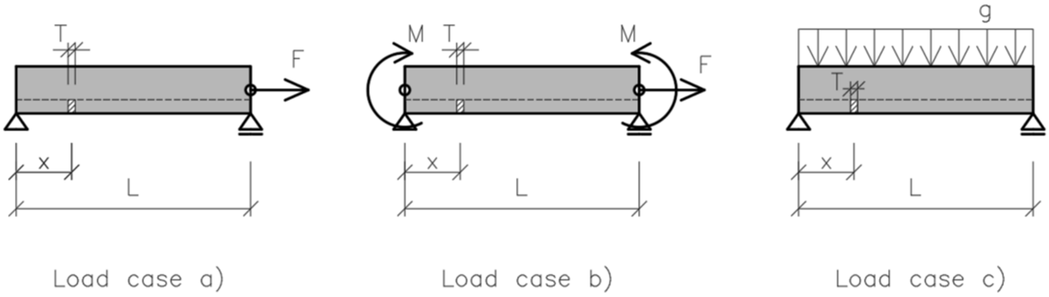

Fig. 1 Load cases used for model demonstration.

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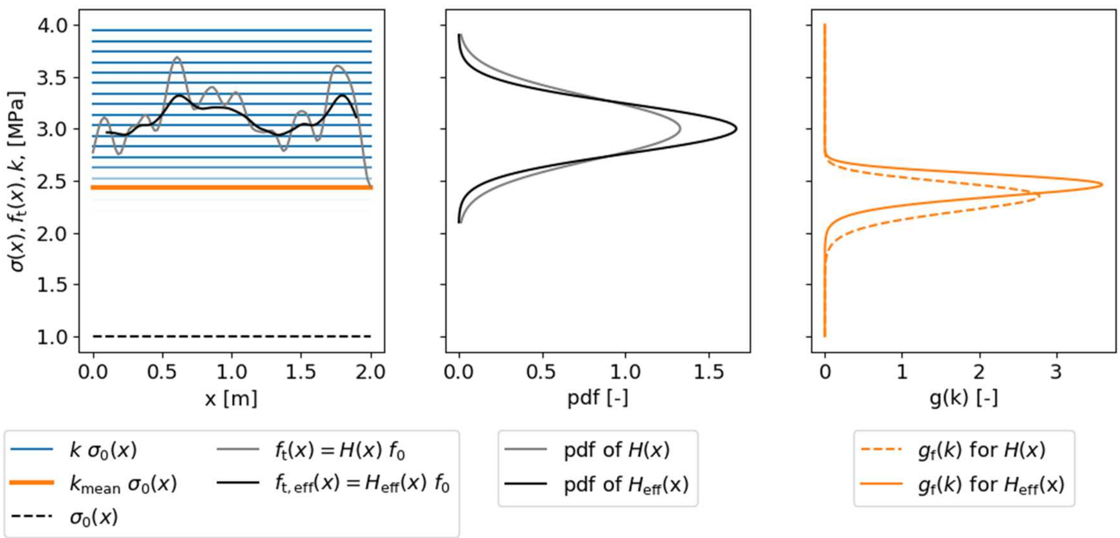
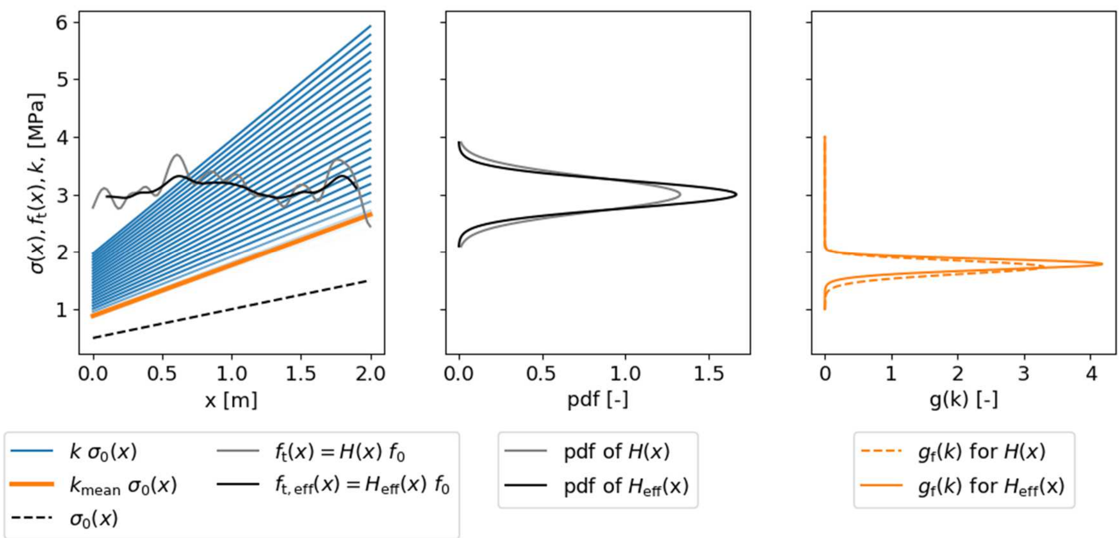
In all cases, the strength field is averaged by a critical volume of length two times larger than the autocorrelation length ($T = 2\ l_\rho$).

Let's look at the Fig. 2. In the left column, the grey and black waved lines show a realization of a varying non-averaged (grey) and averaged (black) strengths $f_0 H(\mathrm{x})$. In the middle column, there are distribution density functions of these processes. For each case, the unit stress function σ_0 was prescribed with a value of 1 in the middle of the specimen L/2 and then scaled by parameter k. The left column shows the unit stress functions as dashed blue lines and several scaled stress functions as blue lines, where darker colors indicate a higher probability that the stress level will by crossed by the strength field. The orange color shows the distribution of stress, where the crossing probability is average. Lastly, the column on the right shows the whole probability density of the stress crossing for scaling parameter k.

From Fig. 1., it is clear that the case with linear stress yields a much lower maximum stress level than the case with constant stress and for the parabolic stress, the contrary. Besides that is clear, that the maximum stress level for non-averaged strength fields is much lower.



Load case a).



Load case b).

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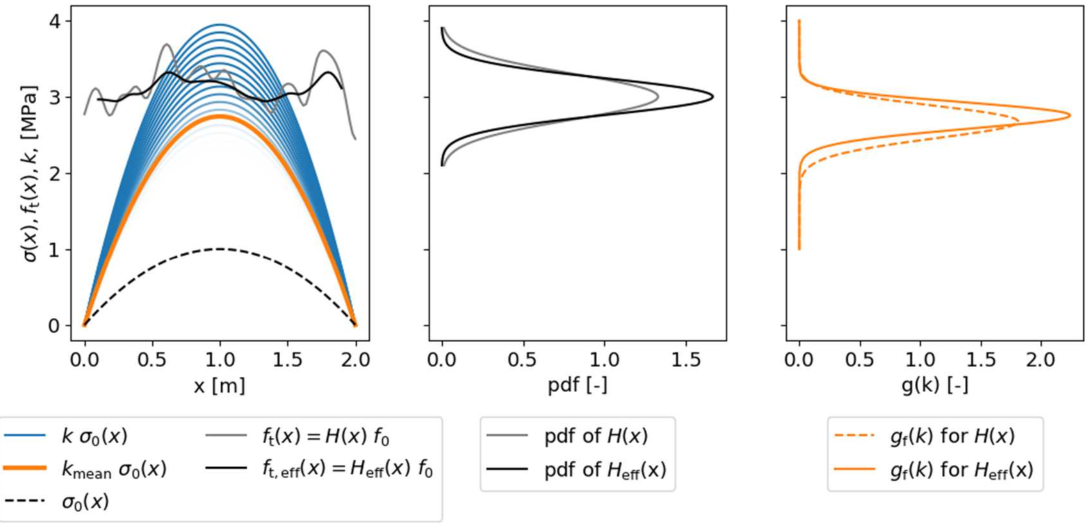

Load case c).

Fig. 2 Probability of stress field crossing random strength.

Size effect scaling

Now, let´s look on the capability of the model to predict structural size effect. For this purpose, we choose the parabolic loading case as shown in Fig. 3. The beam is scaled by adjusting its length L and the unit stress level is scaled such that the stress value at L/2 remains equal to 1. The scaling of stress is shown for lengths of 0.25 and 2 meters. From Fig. 3., it is evident that for a larger specimen, it is more likely that the strength will cross the given stress function. This implies that the level of average maximum stress must be lower than that of a smaller specimen. This serves as a graphical representation of the mechanism of the statistical size effect. If we keep T fixed and plot k_{mean} as a function of specimen length, we get the statistical size effect law for the given bent beam.

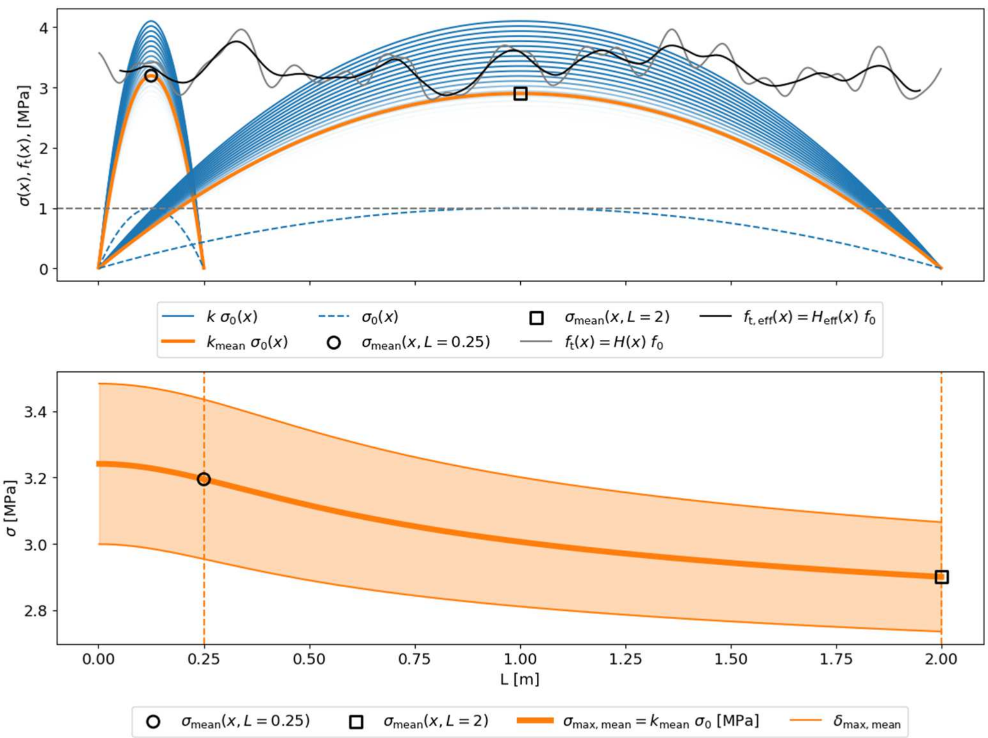

Fig. 3 Nominal strength and specimen length dependence.

4 DISCUSSION

From the results, it is clear that the model is able to represent the correct size effect behavior according to the theory of statistical size effect. In Fig. 2., we can see that the law asymptotically approaches the exponential Weibull theory for large specimen sizes. Furthermore, the response to different stress functions seems to be correct and the averaging of strength through critical volume has the effect of reducing variance and increasing structural strength. Consequently, the model is ready for validation with experimental data and numerical simulations.

5 CONCLUSION

The work presented an analytical model capable of:

- computing the probability distribution function of achieving maximum strength in concrete materials,
- predicting the size effect law for concrete specimens,
- considering the averaging effects of nonlinearities in the fracture process zone.

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References

[1] CUSATIS, Gianluca, BAŽANT, Zdeněk, CEDOLIN, Luigi. Confinement-shear lattice model for concrete damage in tension and compression. I. Theory. *Journal of Engineering Mechanics* [online]. 2003. 129(12):1439–1448 [ISSN 0733-9399]. DOI 10.1061/(ASCE)0733-9399(2003)129:12(1439)

[2] BAŽANT, Zdeněk. Crack band model for fracture of geomaterials, Proc., 4th Intern. Conf. on Numerical Methods in Geomechanics, held in Edmonton, Alberta, Canada, June 1982, Ed. Z. Eisenstein, Vol. 3, pp. 1137–1152

[3] JIRÁSEK, Milan. Nonlocal models for damage and fracture: Comparison of approaches. *International Journal of Solids and Structures* [online]. 1998. 35, 4133–4145. DOI 10.1016/S0020-7683(97)00306-5

[4] BAŽANT, Zdeněk, Qiang YU. Size Effect in Frac-ture of Concrete Specimens and Structures: NewProblems and Progress. In: Acta Polytechnica [online]. 2004. 44.5-6 DOI 10.14311/608

[5] BAŽANT, Zdeněk. Size Effect in Blunt Fracture: Concrete, Rock, Metal. In: *Journal of Engineering Mechanics* [online] 1984.110.4, pp. 518–535. DOI 10.1061/(ASCE)0733-9399

[6] WEIBULL, Waloddi. A statistical theory of the strength of materials, volume (Handlingar Nr.) 151 of Royal Swedish Institute of Engineering Research (Ingeniörsvetenskapsakademiens). Generalstabens litografiska anstalts förlag. 1939. Stockholm.

[7] DITLEVSEN, Ove. Extremes of random fields over arbitrary domains with application to concrete rupture stresses. *Probabilistic Engineering Mechanics*. 2004. 19. 373–384. DOI 10.1016/j.probengmech.2004.04.001

[8] DITLEVSEN, Ove. Extremes of realizations of continuous time stationary stochastic processes on closed intervals. *Journal of Mathematical Analysis and Applications*. 1966;14(3):463–74.. ISSN 0022-247X.

[9] VOŘECHOVSKÝ, Miroslav and Jan ELIÁŠ. Fracture in random quasibrittle media: II. Analytical model based on extremes of averaging process. In: Engineering Fracture Mechanics [online]. 2020. 235, p. 107155. ISSN 0013-7944. DOI 10.1016/j. engfracmech.2020.107155

[10] VANMARCKE Erik. Random fields: Analysis and synthesis. World Scientific Publishing, revised and expanded new edition; 2010. ISBN 978-981-256-297-5. https://doi.org/10.1142/5807

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